On integration of some classes of $(n+1)$ dimensional nonlinear Partial Differential Equations

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Abstract

The paper represents the method for construction of the families of particular solutions to some new classes of $(n+1)$ dimensional nonlinear Partial Differential Equations (PDE). Method is based on the specific link between algebraic matrix equations and PDE. Admittable solutions depend on arbitrary functions of n variables.

1 Introduction

Many different methods have been developed for analytical investigation of nonlinear PDE during last decades. Especially attractive are methods for study of so-called completely integrable systems. The particular interest to these equations is enhanced due to their wide range of application in physics. We emphasize different dressing methods, which are based on fundamental properties of linear operators, either differential or integral: Zakharov-Shabat dressing method [1, 2], $\bar{\partial}$-problem [3, 4, 5, 6], Sato theory [7, 8].

We suggest the method for construction of the families of particular solutions to some new classes of $(n+1)$ dimensional nonlinear PDE, $n \geq 2$. It is based on general properties of linear *algebraic* matrix equations. Essentially we develop some ideas represented in the ref.[7] and recently in the ref. [8, 9]. In general, for $(n+1)$ dimensional PDE this method supplies solutions, depending on the set of arbitrary functions of n variables. The represented method works also for classical $(2+1)$-dimensional PDE integrable by the Inverse Scattering Technique (IST). In this case our algorithm is similar to the algorithm represented in refs.[7, 10].

The structure of the paper is following. First, we discuss general algorithm relating linear algebraic equation with nonlinear PDE. Then we show that these PDE are compatibility condition for some overdetermined linear system of equations having different structure in comparison with linear system associated with completely integrable nonlinear PDE. We give an example of (2+1)-dimensional system which can not be described in frames of classical dressing methods.

2 General algorithm

As mentioned above, our algorithm is based on the fundamental properties of linear matrix algebraic equation

$$\Psi U = \Phi, \tag{1}$$

where $\Psi = \{\psi_{ij}\}$ is $N \times N$ nondegenerate matrix, U and Φ are $N \times M$ matrices $M < N$. Namely, the solution of this equation is unique, $U = \Psi^{-1}\Phi$, and consequently the homogeneous equation with matrix Ψ has only the trivial solution. Thus, if we find transformation T which maps the nonhomogeneous equation (1) into the homogeneous equation $\Psi\tilde{U}(U) = 0$, then $\tilde{U}(U) = 0$.

Let us show that such transformations can be performed by means of differential operators having special structure. For this purpose let us introduce two types of additional parameters $x = (x_1, \dots, x_n)$ ($n = \dim(x)$) and $t = (t_1, t_2, \dots)$ with the following systems:

$$\Psi_{x_i} = \Psi B_i + \Phi C_i, \quad i = 1, \dots, n \tag{2}$$

(B_i and C_i are constant $N \times N$ and $M \times N$ matrices respectively) and

$$\mathcal{M}_i\Psi = 0, \quad \mathcal{M}_i\Phi = 0, \quad \mathcal{M}_i = \partial_{t_i} + L_i, \tag{3}$$

where L_i are arbitrary linear differential operators having derivatives with respect to variables x_j and constant scalar coefficients, so that the system (2) is compatible with the system (3). For the sake of simplicity in this paper we use only one parameter t, omit subscripts in the eq.(3) and use n-dimensional Laplacian for L:

$$\mathcal{M} = \partial_t + \sum_{k=1}^{n} \alpha_k \partial^2_{x_k} \tag{4}$$

Hereafter indexes i, j and k run values from 1 to n unless otherwise specified.

Let us study compatibility conditions for the system (2) itself, which has the following form:

$$(\Psi B_j + \Phi C_j))B_i + \Phi_{x_j} C_i = (\Psi B_i + \Phi C_i))B_j + \Phi_{x_i} C_j. \tag{5}$$

Require that matrices B_i and C_i satisfy two conditions:

$$C_j B_i - C_i B_j = 0, \quad B_j B_i - B_i B_j = 0, \quad i \neq j, \tag{6}$$

and matrices C_j have the following structure: $C_j = \left[\begin{array}{c|c} P_j & 0_{M,N-R} \end{array}\right]$, $R \leq M$, where P_j are $M \times R$ matrices with rang R and $0_{A,B}$ means $A \times B$ zero matrix. Then equation (5) is reduced to the next one:

$$\Phi_{x_i} P_j - \Phi_{x_j} P_i = 0, \tag{7}$$

which results in the first nonlinear equation for U owing to the eq.(1):

$$(B_j + UC_j)UP_i + U_{x_j} P_i = (B_i + UC_i)UP_j + U_{x_i} P_j. \tag{8}$$

Let us show that another nonlinear matrix equation can be derived using operator $\mathcal{M}$. For this purpose we apply operator $\mathcal{M}$ to both sides of the eq. (1) and use eqs. (2) and (3):

$$0 = \mathcal{M}\Phi = (\mathcal{M}\Psi)U + \Psi\mathcal{M}U + 2\sum_{k=1}^{n} \alpha_k \Psi_{x_k} U_{x_k} = \Psi\left(\mathcal{M}U + 2\sum_{k=1}^{n} \alpha_i (B_k + UC_k) U_{x_k}\right) = 0 \quad (9)$$

Since $\det(\Psi) \neq 0$ one has the second nonlinear equation for the matrix U:

$$U_t + \sum_{k=1}^{n} \alpha_k \left(U_{x_k x_k} + 2(B_k + UC_k) U_{x_k}\right) = 0. \quad (10)$$

Having eqs.(6,8,10) one can derive the complete system of equations for elements of the matrix V composed of first R rows of the matrix U. First equation exists if $n > 2$. In this case we can write the matrix equation without operators B_i using any three equations (8) with pairs of indexes (i, j), (j, k) and (k, i) and relations (6) ($U_i = VP_i$):

$$\sum_{perm} P_i \left(U_{j_{x_k}} - U_{k x_j} + [U_k, U_j]\right) = 0, \quad (11)$$

where sum is over cycle permutation of indexes i, j and k. In general, this equation is not complete system for all elements of the matrix V. To complete it we derive additional matrix equation, which can be done for any n. Eliminate operators B_i from the eq. (8) ($i = 1$, $j = 2$) using eqs. (6) and (10):

$$P_i U_{j_t} - P_j U_{it} + 2\sum_{k=1, k\neq i}^{n} \alpha_k (P_i U_k - P_k U_i) U_{j_{x_k}} - 2\sum_{k=1, k\neq j}^{n} \alpha_k (P_j U_k - P_k U_j) U_{i x_k} + \quad (12)$$

$$\sum_{k=1}^{n} \alpha_k \left(P_i U_{j_{x_k x_k}} - P_j U_{i x_k x_k} + 2P_k (U_{i x_j} - U_{j_{x_i}})_{x_k} - 2P_k (U_{i x_k} U_j - U_{j_{x_k}} U_i)\right) = 0$$

In particular, if $\alpha_j = 0$, $j > 1$, $\alpha_1 = 1$, then eq.(12) reduces in

$$P_1 \left(U_{2t} + 2U_{1 x_1 x_2} - U_{2 x_1 x_1} + 2(U_2 U_1)_{x_1} - 2U_{1 x_1} U_2\right) - \quad (13)$$
$$P_2 \left(U_{1t} + U_{1 x_1 x_1} + 2U_1 U_{1 x_1}\right) = 0.$$

Thus equations (11,12,13) do not depend on both parameter N (which characterizes dimensions of the matrices in the eq.(1)) and matrices B_i, i.e. N is arbitrary positive integer, B_i are arbitrary $N \times N$ matrices fitting relations (6).

We point on two trivial reductions of the eq.(13).

1. Let matrix A exist such that $AP_1 = 0$ and $AP_2 = I_R$ (I_R is $R \times R$ identity matrix). Multiplying eq.(13) by A from the right one receives matrix Burgers equation for U_1.

2. If $R = M = 2$ and

$$P_1 = I_2, \quad P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}, \quad (14)$$

then eq.(13) is reduced to the next system:

$$r_t - r_{x_1 x_2} - 2r w_{x_1 x_2} = 0, \quad q_t + q_{x_1 x_2} + 2q w_{x_1 x_2} = 0, \quad w_{x_1 x_1} - w_{x_2 x_2} = qr, \quad (15)$$

where functions r and q are related with elements of the matrix V by the formulae

$$u_1 = \frac{1}{4}(r - q + 2w_{x_1}), \quad u_2 = \frac{1}{4}(r + q + 2w_{x_2}), \qquad (16)$$
$$v_1 = \frac{1}{4}(-r - q + 2w_{x_2}), \quad v_2 = \frac{1}{4}(-r + q + 2w_{x_1}).$$

Eq.(15) becomes Devi-Stewartson equation (DS) after reduction $r = \psi$, $q = \bar{\psi}$, $t_1 = it$, where $i^2 = -1$, bar means complex conjugated value.

We will see in the next section, that the case $R = M$ does always correspond to the classical completely integrable $(2 + 1)$-dimensional systems.

Arbitrary functions of variables x_j ($j = 1, \ldots, n$) appear in the solution V due to the matrix function Φ, defined by the system (7). The number of arguments in the arbitrary functions as well as the number of these functions is defined by particular choice of the matrices P_j and dimension n of x-space. If $n = 2$, then one has at most N arbitrary functions of two variables, see **Example**. In general, for n-dimensional x-space and $R < M$ we are able to represent examples with N functions of n variables. If $R = M$ then Φ may depend at most on $N \times M$ arbitrary scalar functions of *single* variable, which is in accordance with [7]. Detailed discussion of this problem is left beyond the scope of this paper.

2.1 On the operator representation of PDE

We derive the overdetermined linear system of PDE with compatibility condition in the form of eqs.(11) and (12). First, introduce arbitrary $R \times N$ matrix function $\mathbf{R}(\lambda)$ of the additional parameter λ. Multiply eqs.(8) and (10) by $\mathbf{R}(\lambda)\exp(\eta)$ from the left and introduce function $\hat{\Psi} = \mathbf{R}e^{\eta}U$

$$\eta = \sum_{k=1}^{n} B_k x_k - \left(\sum_{k=1}^{n} \alpha_k B_k^2\right) t. \qquad (17)$$

We get after transformations:

$$\hat{\Psi}_{x_j} P_i - \hat{\Psi}_{x_i} P_j = \hat{\Psi} P_i U_j - \hat{\Psi} P_j U_i, \qquad (18)$$
$$\hat{\Psi}_t + \sum_{k=1}^{n} \alpha_k \left(\hat{\Psi}_{x_k x_k} + 2\hat{\Psi} P_k V_{x_k}\right) = 0. \qquad (19)$$

If $R = M$, t.e. all P_j are square nondegenerate matrices, then the system (18), (19) is equivalent to the classical $M \times M$ overdetermined linear system for correspondent (2+1)-dimensional integrable system. In fact, one can express all derivatives of $\hat{\Psi}$ with respect to x_j, $j > 1$ through the derivatives of $\hat{\Psi}$ with respect to x_1 using equation (18). Both equations (18) and (19) are $M \times M$ matrix equations for $M \times M$ matrix function $\hat{\Psi}$. Thus eq. (18) can be taken for the spectral problem while eq. (19) represents evolution part of the overdetermined linear system. For instance, if $M = R = n = 2$, $\alpha_1 = 1$, $\alpha_k = 0$, $k > 1$, and P_i have the form (14), then the compatibility condition of the linear system (18,19) is given by the eqs.(15).

In the case $R < M$ situation is different. $\hat{\Psi}$ is $R \times M$ matrix function, while (18) is $R \times R$ matrix equation. Thus it can not be taken for the spectral problem. Also, eq.(19) involves all derivatives which form operator M. So, we have $(n + 1)$ dimensional equations. Below we represent example of $(2 + 1)$-dimensional system of this type.

3 Example

Let $M = 3$, $R = 2$, $Q = 2$, $\alpha_2 = \alpha_3 = 0$, $\alpha_1 = 1$, $N = 3k + 2$, $k = 1, 2, \ldots$. Everywhere indexes i and j take values 1 and 2. Let

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix},$$

$$B_j = \begin{bmatrix} 0_{2,2} & b_{j1} & 0_{2,N-5} \\ 0_{3(k-1),2} & 0_{3(k-1),3} & b_{j2} \\ 0_{3,2} & 0_{3,3} & 0_{3,N-5} \end{bmatrix}, b_{11} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$b_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad b_{j2} = \mathrm{diag}(\underbrace{A_j, A_j, \ldots}_{k-1}), \quad j = 1, 2,$$

$$A_1 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

$\Phi = [\phi_1 \quad \phi_2 \quad \phi_3]$, where ϕ_k are N-dimensional columns. Then equation (7) can be written in the form $\phi_{1x_2} = \phi_{2x_1}$, $\phi_{2x_2} = \phi_{3x_1}$, i.e. $\phi_1 = S_{x_1x_1}$, $\phi_2 = S_{x_1x_2}$, $\phi_3 = S_{x_2x_2}$, where S is arbitrary function of variables x_1 and x_2. In view of eq.(3), we can write for Φ: $\Phi = \int_{-\infty}^{\infty} c(k_1, k_2)[1 \quad k_2/k_1 \quad k_2^2/k_1^2] \exp[k_1x_1 + k_2x_2 - k_1^2 t] dk_1 dk_2$, where $c(k_1, k_2)$ is arbitrary column of N elements. Function Ψ solves system (2,3) and can be represented in the form $\Psi = (\Psi_0 + \tilde{\Psi})e^{\eta}$, where η is given by the eq.(17), $\tilde{\Psi} = \partial_{x_1}^{-1} \Phi C_1 e^{-\eta}$ and Ψ_0 is arbitrary constant $N \times N$ matrix. Then eq.(1) can be solved for the function U, its first R rows define matrix V. Elements of this matrix satisfy the system (13) which gets the following form:

$$\begin{aligned}
u_{1t} + u_{1x_1x_1} - 2u_{2x_1x_2} + 2v_{2x_1x_1} + 2u_1u_{1x_1} - 2(v_2u_1)_{x_1} + \\
2v_1u_{2x_1} - 2(w_2u_2)_{x_1} + 2(u_2v_1)_{x_1} + 4v_2v_{2x_1} &= 0 \\
v_{1t} - v_{x_1x_1} + 2u_{1x_1x_2} + 2(w_1u_2)_{x_1} + 2u_1v_{1x_1} - 2v_2v_{1x_1} &= 0 \\
w_{1t} - w_{1x_1x_1} + 2v_{1x_1x_2} - 2w_1u_{1x_1} + 4v_1v_{1x_1} - 2w_2v_{1x_1} + 2(w_1v_2)_{x_1} &= 0 \\
u_{2t} + u_{2x_1x_1} + 2u_2u_{1x_1} + 2v_2u_{2x_1} = 0, \quad v_{2t} + v_{2x_1x_1} + 2u_2v_{1x_1} + 2v_2v_{2x_1} &= 0 \\
w_{2t} - w_{2x_1x_1} + 2u_{1x_1x_2} + 2v_{2x_1x_2} - 2v_{1x_1x_1} + 2u_2w_{1x_1} + 2v_2w_{2x_1} &= 0
\end{aligned}$$

4 Conclusions

The represented version of the dressing method serves for wide class of $(n+1)$-dimensional PDE. It supplies solutions depending on arbitrary functions of n variables provided $R < M$. In the last case solution depends on the functions of single variables. By construction, equations have infinite number of commuting flows and are compatibility conditions for some specific linear overdetermined systems, which is equivalent to the classical linear problem if only $R = M$.

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